

# ANALYTIC SOLUTIONS FOR THE APPROXIMATED 1-D KANTOROVICH MASS TRANSFER PROBLEMS

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**ABSTRACT.** This paper mainly investigates the approximation of a global maximizer of the 1-D Monge-Kantorovich mass transfer problem through the approach of non-linear differential equations with Dirichlet boundary. Using an approximation mechanism, the primal maximization problem can be transformed into a sequence of minimization problems. By applying the canonical duality theory, one is able to derive a sequence of analytic solutions for the minimization problems. In the final analysis, the convergence of the sequence to a global maximizer of the primal Monge-Kantorovich problem will be demonstrated.

## 1. INTRODUCTION

Mass transfer is the net movement of mass from one location to another by the action of driving forces, such as pressure gradient (pressure diffusion), temperature gradient (thermal diffusion), etc. In our physical world, when a system contains more components with various concentration from point to point, a natural tendency for mass to be transferred occurred in order to minimize any concentration difference within the system. This transfer phenomenon is governed by Fick's First Law. The original transfer problem, which was proposed by Monge [15], investigated how to move one mass distribution to another one with the least amount of work by searching for a mapping  $\mathbf{s}$  to minimize the cost functional

$$C[\mathbf{r}] := \int_{\Omega} |x - \mathbf{r}(x)| d\mu^+(x)$$

among the 1-1 mappings  $\mathbf{r} : \Omega \rightarrow \Omega^*$  that push forward  $\mu^+$  into  $\mu^-$ , where both  $\Omega$  and  $\Omega^*$  are bounded domains in  $\mathbb{R}^n$ ,  $\mu^+$  and  $\mu^-$  are two nonnegative Radon measures on  $\Omega$  and  $\Omega^*$ , respectively.

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Keywords: Monge-Kantorovich mass transfer, singular variational problem, canonical duality theory.

Mathematics Subject Classification: 35J20, 35J60, 49K20, 80A20.

In the 1940s, Kantorovich initiated a duality theory by relaxing Monge's transfer problem to the task of finding a global maximizer (so-called *Kantorovich potential*) for the Kantorovich problem in the following form [12, 13],

$$(1) \quad (\mathcal{P}) : \max_u \left\{ K[u] := \int_U u f dx \right\},$$

where  $U = \Omega \cup \Omega^*$ ,  $f := f^+ - f^-$ ,  $f^+ \in C(\overline{\Omega})$  and  $f^- \in C(\overline{\Omega^*})$  are two nonnegative density functions and satisfy the normalized balance condition

$$\int_{\Omega} f^+ dx = \int_{\Omega^*} f^- dx = 1.$$

$u$  is subject to the following constraints,

$$(2) \quad u \in W_0^{1,\infty}(U) \cap C(\overline{U}),$$

$$(3) \quad u = 0 \text{ on } \overline{\Omega \cap \Omega^*},$$

$$(4) \quad \|u_x\|_{L^\infty(U)} \leq 1.$$

In particular, when  $\Omega \cap \Omega^* = \emptyset$ ,  $C(\overline{U})$  represents  $C(\overline{\Omega})$  and  $C(\overline{\Omega^*})$ , respectively.

As a matter of fact, the Kantorovich problem may not be a perfect dual to the Monge problem unless a so-called *dual criteria for optimality* is satisfied [4, 5]. Nowadays, the Monge-Kantorovich mass transfer model is widely used in the diffusive and convective transport of chemical species, purification of blood in the kidneys and livers, separation of chemical components in distillation columns, controlling haze in the atmosphere by artificial precipitation, etc. Interested readers can refer to [1, 2, 5, 12, 13, 15, 17] for more important applications.

Indeed, many mathematical tools have been developed for the infinite-dimensional linear programming [16, 17, 18], etc. In this paper, we consider the 1-D Monge-Kantorovich problem (1) through a nonlinear differential equation approach by introducing a sequence of approximation problems for the primal problem  $(\mathcal{P})$ ,

$$(5) \quad (\mathcal{P}^{(k)}) : \min_{w_k} \left\{ I^{(k)}[w_k] := \int_U L^{(k)}(w_{k,x}, w_k, x) dx = \int_U \left( H^{(k)}(w_{k,x}) - w_k f \right) dx \right\},$$

where  $w_{k,x}$  is the weak derivative of  $w_k$  with respect to  $x$ ,  $H^{(k)} : \mathbb{R} \rightarrow \mathbb{R}^+$  is defined as

$$H^{(k)}(x) := e^{k(x^2-1)/2}/k.$$

In particular,  $I^{(k)}$  is called the *potential energy functional* and is weakly lower semicontinuous on  $W_0^{1,\infty}(U)$ . Moreover,  $L^{(k)}(P, z, x) : \mathbb{R} \times \mathbb{R} \times U \rightarrow \mathbb{R}$  satisfies the following coercivity inequality and is convex in the variable  $P$ ,

$$L^{(k)}(P, z, x) \geq p_k P^2 - q_k, \quad P \in \mathbb{R}, z \in \mathbb{R}, x \in U,$$

for certain constants  $p_k$  and  $q_k$ . Notice that when  $|x| \leq 1$ , then  $\lim_{k \rightarrow \infty} H^{(k)}(x) = 0$  uniformly. All these facts assures the existence of the global minimizer for  $\mathcal{P}^{(k)}$  [8]. Once such a sequence of global minimizers  $\{\bar{u}_k\}_k$  is obtained, then it will help find a global Kantorovich potential which solves the primal problem  $(\mathcal{P})$ , as is explained in

Theorem 1.2.

The key mission of this paper is to obtain an explicit representation of this approximation sequence  $\{\bar{u}_k\}_k$ . By variational calculus, one derives a correspondingly sequence of Euler-Lagrange equations for  $(\mathcal{P}^{(k)})$ , namely, for any  $k \in \mathbb{N}$ ,

$$(6) \quad (e^{k(u_{k,x}^2-1)/2} u_{k,x})_x + f = 0, \quad \text{in } U \setminus \{\overline{\Omega \cap \Omega^*}\},$$

equipped with the Dirichlet boundary condition. The term  $e^{k(\nabla u_k^2-1)/2}$  is called the *transport density*. Actually, (6) is a highly nonlinear differential equation which is difficult to solve by the direct approach [3, 8, 14]. However, by the canonical duality theory, one is able to demonstrate the existence and uniqueness of the solution for the nonlinear differential equation, which establishes the equivalence between the global minimizer of  $(\mathcal{P}^{(k)})$  and the solution of Euler-Lagrange equation (6).

In the former literature, as far as we know, few authors considered the analytic Kantorovich potential of the approximation problems  $(\mathcal{P}^{(k)})$ , either a-priori estimates or numerical approach [5, 6, 7, 16]. The purpose of this paper is to investigate the analytic solutions for the optimization problems  $(\mathcal{P}^{(k)})$  through *canonical duality theory* introduced by G. Strang et al. [9]. These methods were originally proposed to find global minimizers for a non-convex strain energy functional with a double-well potential, which was a very challenging nonconvex problem.

At the moment, we would like to introduce the main theorems.

**Theorem 1.1.** *For any positive density functions  $f^+ \in C(\overline{\Omega})$  and  $f^- \in C(\overline{\Omega^*})$  satisfying the normalized balance condition, there exists a sequence of solutions  $\{\bar{u}_k\}_k$  subject to the constraints (2)-(4) for the Euler-Lagrange equations (6), which is at the same time a sequence of global minimizers for the approximation problems  $(\mathcal{P}^{(k)})$ . In particular, when  $\Omega = (a, b)$ ,  $\Omega^* = (c, d)$ ,  $\Omega \cap \Omega^* = \emptyset$ , then  $\{\bar{u}_k\}_k$  can be represented explicitly as*

$$\bar{u}_k(x) = \begin{cases} \int_a^x (-F^+(t) + C_k)/E_k^{-1}((-F^+(t) + C_k)^2)dt, & x \in [a, b], \\ \int_c^x (F^-(t) - D_k)/E_k^{-1}((F^-(t) - D_k)^2)dt, & x \in [c, d], \end{cases}$$

where  $E_k$ ,  $F$  and  $G$  are defined as

$$\begin{cases} E_k(x) := x^2 \ln(ex^{2/k}), & x \in (0, 1], \\ F^+(x) := \int_a^x f^+(t)dt, & x \in [a, b], \\ F^-(x) := \int_c^x f^-(t)dt, & x \in [c, d], \end{cases}$$

$E_k^{-1}$  stands for the inverse of  $E_k$ , both  $\{C_k\}_k$  and  $\{D_k\}_k$  are number sequences in  $(0, 1)$ .

By Rellich-Kondrachov Compactness Theorem, we have the following convergence result.

**Theorem 1.2.** *For any positive density functions  $f^+ \in C(\overline{\Omega})$  and  $f^- \in C(\overline{\Omega}^*)$  satisfying the normalized balance condition, there exists a global minimizer subject to (2)-(4) for the Kantorovich problem  $(\mathcal{P})$ .*

The rest of the paper is organized as follows. In Section 2, first, we introduce some useful notations which will simplify our proof considerably. Then, we apply the canonical dual transformation to deduce a sequence of perfect dual problems  $(\mathcal{P}_d^{(k)})$  corresponding to  $(\mathcal{P}^{(k)})$  and a pure complementary energy principle. Next, we apply the canonical duality theory to prove Theorem 1.3. In the final analysis, a global maximizer of the primal Monge-Kantorovich problem will be given by the approximation techniques in the proof of Theorem 1.4.

## 2. PROOF OF THE MAIN RESULTS

### 2.1. Some useful notations.

- $\theta_k$  is given by

$$\theta_k(x) := e^{k(w_{k,x}^2 - 1)/2} w_{k,x}.$$

- $\Phi^{(k)}$  is a nonlinear geometric mapping defined as

$$\Phi^{(k)}(w_k) := k(w_{k,x}^2 - 1)/2.$$

For convenience's sake, denote  $\xi_k := \Phi^{(k)}(w_k)$ . It is evident that  $\xi_k$  belongs to the function space  $\mathcal{U}$  given by

$$\mathcal{U} := \left\{ \phi \in L^\infty(U) \mid \phi \leq 0 \right\}.$$

- $\Psi^{(k)}$  is a canonical energy defined as

$$\Psi^{(k)}(\xi_k) := e^{\xi_k}/k,$$

which is a convex function with respect to  $\xi_k$ . For simplicity, denote  $\zeta_k := \Psi^{(k)}(\xi_k)$ , which is the Gâteaux derivative of  $\Psi^{(k)}$  with respect to  $\xi_k$ . Moreover,  $\zeta_k$  is invertible with respect to  $\xi_k$  and belongs to the function space  $\mathcal{V}^{(k)}$ ,

$$\mathcal{V}^{(k)} := \left\{ \phi \in L^\infty(U) \mid 0 < \phi \leq 1/k \right\}.$$

- $\Psi_*$  is defined as

$$\Psi_*(\zeta_k) := \xi_k \zeta_k - \Psi^{(k)}(\xi_k) = \zeta_k (\ln(k\zeta_k) - 1).$$

- $\lambda_k$  is defined as  $\lambda_k := k\zeta_k$ , and belongs to the function space  $\mathcal{V}$ ,

$$\mathcal{V} := \left\{ \phi \in L^\infty(U) \mid 0 < \phi \leq 1 \right\}.$$

**2.2. Proof of Theorem 1.1.** Before we prove the main result, first and foremost, we give some useful definitions.

**Definition 2.1.** *By Legendre transformation, one defines a total complementary energy functional  $\Xi^{(k)}$ ,*

$$\Xi^{(k)}(u_k, \zeta_k) := \int_U \left\{ \Phi^{(k)}(u_k) \zeta_k - \Psi_*^{(k)}(\zeta_k) - f u_k \right\} dx.$$

Next we introduce an important *criticality criterium* for the total complementary energy functional.

**Definition 2.2.**  $(\bar{u}_k, \bar{\zeta}_k)$  is called a critical pair of  $\Xi^{(k)}$  if and only if

$$(7) \quad D_{u_k} \Xi^{(k)}(\bar{u}_k, \bar{\zeta}_k) = 0,$$

and

$$(8) \quad D_{\zeta_k} \Xi^{(k)}(\bar{u}_k, \bar{\zeta}_k) = 0,$$

where  $D_{u_k}, D_{\zeta_k}$  denote the partial Gâteaux derivatives of  $\Xi^{(k)}$ , respectively.

In effect, by variational calculus, we have the following observations from (7) and (8).

**Lemma 2.3.** *On the one hand, for any fixed  $\zeta_k \in \mathcal{V}^{(k)}$ , (7) is equivalent to the equilibrium equation*

$$(k\zeta_k \bar{u}_{k,x})_x + f = 0, \quad \text{in } U \setminus \{\overline{\Omega \cap \Omega^*}\}.$$

*On the other hand, for any fixed  $u_k$  subject to (2)-(4), (8) is consistent with the constructive law*

$$\Phi^{(k)}(u_k) = D_{\zeta_k} \Psi_*^{(k)}(\bar{\zeta}_k).$$

Lemma 2.3 indicates that  $\bar{u}_k$  from the critical pair  $(\bar{u}_k, \bar{\zeta}_k)$  solves the Euler-Lagrange equation (7).

**Definition 2.4.** *From Definition 2.1, one defines a pure complementary energy  $I_d^{(k)}$  in the form*

$$I_d^{(k)}[\zeta_k] := \Xi^{(k)}(\bar{u}_k, \zeta_k),$$

where  $\bar{u}_k$  solves the Euler-Lagrange equation (6).

For convenience's sake, we show another representation of the pure energy  $I_d^{(k)}$ .

**Lemma 2.5.** *The pure complementary energy functional  $I_d^{(k)}$  can be rewritten as*

$$I_d^{(k)}[\zeta_k] = -1/2 \int_U \left\{ |\theta_k|^2 / (k\zeta_k) + k\zeta_k + 2\zeta_k (\ln(k\zeta_k) - 1) \right\} dx,$$

where  $\theta_k$  satisfies

$$(9) \quad \theta_{k,x} + f = 0 \quad \text{in } U \setminus \{\overline{\Omega \cap \Omega^*}\},$$

equipped with a hidden boundary condition.

*Proof.* Through integrating by parts, one has

$$\begin{aligned} I_d^{(k)}[\zeta_k] &= - \underbrace{\int_U \left\{ (k\zeta_k \bar{u}_{k,x})_x + f \right\} \bar{u}_k dx}_{(I)} \\ &\quad - \underbrace{1/2 \int_U \left\{ k\zeta_k \bar{u}_{k,x}^2 + k\zeta_k + 2\zeta_k(\ln(k\zeta_k) - 1) \right\} dx}_{(II)}. \end{aligned}$$

Since  $\bar{u}_k$  solves the Euler-Lagrange equation (6), then, the first part (I) disappears. Keeping in mind the definition of  $\theta_k$  and  $\zeta_k$ , one reaches the conclusion.  $\square$

With the above discussion, next we establish a sequence of dual variational problems corresponding to the approximation problems  $(\mathcal{P}^{(k)})$ .

$$(10) \quad (\mathcal{P}_d^{(k)}) : \max_{\zeta_k \in \mathcal{V}^{(k)}} \left\{ I_d^{(k)}[\zeta_k] = -1/2 \int_U \left\{ \theta_k^2 / (k\zeta_k) + k\zeta_k + 2\zeta_k(\ln(k\zeta_k) - 1) \right\} dx \right\}.$$

In effect, by calculating the Gâteaux derivative of  $I_d^{(k)}$  with respect to  $\zeta_k$ , we have

**Lemma 2.6.** *The variation of  $I_d^{(k)}$  with respect to  $\zeta_k$  leads to the Dual Algebraic Equation (DAE), namely,*

$$(11) \quad \theta_k^2 = k\bar{\zeta}_k^2(2\ln(k\bar{\zeta}_k) + k),$$

where  $\bar{\zeta}_k$  is from the critical pair  $(\bar{u}_k, \bar{\zeta}_k)$ .

As a matter of fact, the identity (11) can be rewritten as

$$(12) \quad \theta_k^2 = E_k(\lambda_k) = \lambda_k^2 \ln(e\lambda_k^{2/k}).$$

It is easy to check,  $E_k$  is strictly increasing with respect to  $\lambda \in [e^{-k/2}, 1]$ .

From the above discussion, one deduces that, once  $\theta_k$  is given, then the analytic solution of the Euler-Lagrange equation (6) can be represented as

$$(13) \quad \bar{u}_k(x) = \int_{x_0}^x \eta_k(t) dt,$$

where  $x \in \bar{U}$ ,  $x_0 \in \partial U$ ,  $\eta_k = \theta_k / \lambda_k$ . Next, we verify that  $\bar{u}_k$  is subject to (2)-(4) and is exactly a global minimizer for  $(\mathcal{P}^{(k)})$  and  $\bar{\zeta}_k$  is a global maximizer over  $\mathcal{V}^{(k)}$  for  $(\mathcal{P}_d^{(k)})$ .

**Lemma 2.7.** *(Canonical duality theory) For any positive density functions  $f^+ \in C(\bar{\Omega})$  and  $f^- \in C(\bar{\Omega}^*)$  satisfying the normalized balance condition, there exists a unique sequence of solutions  $\{\bar{u}_k\}_k$  subject to (2)-(4) for the Euler-Lagrange equations (6) with Dirichlet boundary in the form of (13), which is a unique sequence of global minimizers for the approximation problems  $(\mathcal{P}^{(k)})$ . And the corresponding  $\{\bar{\zeta}_k\}_k$  is a unique*

sequence of global maximizers for the dual problems  $(\mathcal{P}_d^{(k)})$ . Moreover, the following duality identity holds,

$$(14) \quad I^{(k)}[\bar{u}_k] = \min_{u_k} I^{(k)}[u_k] = \Xi^{(k)}(\bar{u}_k, \bar{\zeta}_k) = \max_{\zeta_k} I_d^{(k)}[\zeta_k] = I_d^{(k)}[\bar{\zeta}_k].$$

**Remark 2.8.** Lemma 2.7 demonstrates that the maximization of the pure complementary energy functional  $I_d^{(k)}$  is perfectly dual to the minimization of the potential energy functional  $I^{(k)}$ . In effect, the identity (14) indicates there is no duality gap between them.

*Proof.* Without loss of generality, we consider the disjoint case  $\Omega = (a, b)$  and  $\Omega^* = (c, d)$ ,  $b < c$ . We divide our proof into three parts. In the first and second parts, we discuss the uniqueness of  $\theta_k$ . Extremum conditions will be illustrated in the third part.

*First Part:*

In  $\Omega$ , we have a general solution for the differential equation (9) in the form of

$$\theta_k(x) = -F^+(x) + C_k, \quad x \in [a, b].$$

Since  $f^+ > 0$ , then  $F^+ \in C[a, b]$  is a strictly increasing function with respect to  $x \in [a, b]$  and consequently is invertible. From the identity (12), one sees that there exists a unique continuous function  $\lambda_k(x) \in [e^{-k/2}, 1]$ . By paying attention to the Dirichlet boundary  $\bar{u}_k(a) = 0$ , one has the analytic solution  $\bar{u}_k$  in the following form,

$$\bar{u}_k(x) = \int_a^x \eta_k(x) dx, \quad x \in [a, b].$$

Since

$$\lim_{x \rightarrow F^{-1}(C_k)} \eta_k(x) = 0,$$

thus,  $\bar{u}_k \in C[a, b]$ . Recall that

$$\bar{u}_k(b) = \int_a^{F^{-1}(C_k)} \eta_k(x) dx + \int_{F^{-1}(C_k)}^b \eta_k(x) dx = 0,$$

and we can determine the constant  $C_k \in (0, 1)$  uniquely. Indeed, let

$$\mu_k(x, t) := (-F^+(x) + t)/\lambda_k(x, t)$$

and

$$M_k(t) := \int_a^b \mu_k(x, t) dx,$$

where  $\lambda_k(x, t)$  is from (12). It is evident that  $\lambda_k$  depends on  $C_k$ . As a matter of fact,  $M_k$  is strictly increasing with respect to  $t \in (0, 1)$ , which leads to

$$C_k = M_k^{-1}(0).$$

*Second Part:*

Applying the similar procedure, one sees that

$$\theta_k(x) = F^-(x) - D_k, \quad x \in [c, d],$$

where the constant  $D_k \in (0, 1)$ . Since  $f^- > 0$ , then  $F^- \in C[c, d]$  is a strictly increasing function with respect to  $x \in [c, d]$  and consequently is invertible. We can represent the analytical solution  $\bar{u}_k$  in the following form,

$$\bar{u}_k(x) = \int_c^x \eta_k(x) dx, \quad x \in [c, d],$$

Since

$$\lim_{x \rightarrow G^{-1}(D_k)} \eta_k(x) = 0,$$

thus,  $\bar{u}_k \in C[c, d]$ . Recall that

$$\bar{u}_k(d) = 0,$$

and we can determine the constant  $D_k \in (0, 1)$  uniquely. Indeed, let

$$\rho_k(x, t) := (F^-(x) - t)/\lambda_k(x, t)$$

and let

$$N_k(t) := \int_c^d \rho_k(x, t) dx,$$

where  $\lambda_k(x, t)$  is from (12). As a matter of fact,  $N_k$  is strictly decreasing with respect to  $t \in (0, 1)$ , which leads to

$$D_k = N_k^{-1}(0).$$

Furthermore, the other cases, such as  $b = c$  and  $b > c$  can also be discussed similarly due to the fact that  $\bar{u}_k = 0$  on  $\overline{\Omega \cap \Omega^*}$ . Therefore,  $\theta_k$  is uniquely determined in  $U$  and the analytic solution  $\bar{u}_k \in C(\overline{U})$ .

### *Third Part:*

On the one hand, for any test function  $\phi \in W_0^{1,\infty}$ , the second variational form  $\delta_\phi^2 I^{(k)}$  with respect to  $\phi$  is equal to

$$(15) \quad \int_U e^{k(\bar{u}_{k,x}^2 - 1)/2} \left\{ k(\bar{u}_{k,x} \phi_x)^2 + \phi_x^2 \right\} dx.$$

On the other hand, for any test function  $\psi \in \mathcal{V}^{(k)}$ , the second variational form  $\delta_\psi^2 I_d^{(k)}$  with respect to  $\psi$  is equal to

$$(16) \quad - \int_U \left\{ \theta_k^2 \psi^2 / (k \zeta_k^3) + \psi^2 / \zeta_k \right\} dx.$$

From (15) and (16), one deduces immediately that

$$\delta_\phi^2 I^{(k)}(\bar{u}_k) \geq 0, \quad \delta_\psi^2 J_d^{(k)}(\bar{\zeta}_k) \leq 0.$$

Together with the uniqueness of  $\theta_k$  discussed in the first and second parts, the proof is concluded.  $\square$

Consequently, we reach the conclusion of Theorem 1.1 by summarizing the above discussion.



**2.3. Proof of Theorem 1.2.** Now we consider the convergence of the sequence  $\{\bar{u}_k\}_k$  in Theorem 1.1. According to Rellich-Kondrachov Compactness Theorem, since

$$\sup_k |\bar{u}_k| \leq \text{diam}(U)$$

and

$$\sup_k |\bar{u}_{k,x}| \leq 1,$$

then, there exists a subsequence (without any confusion, we still denote as)  $\{\bar{u}_k\}_k$  and  $u \in W_0^{1,\infty}(U) \cap C(\bar{U})$  such that

$$(17) \quad \bar{u}_k \rightarrow u \ (k \rightarrow \infty) \text{ in } L^\infty(U),$$

$$(18) \quad \bar{u}_{k,x} \rightharpoonup^* u_x \ (k \rightarrow \infty) \text{ weakly } * \text{ in } L^\infty(U).$$

From (18), one has

$$\|u_x\|_{L^\infty(U)} \leq \liminf_{k \rightarrow \infty} \|\bar{u}_{k,x}\|_{L^\infty(U)} \leq \sup_{k \rightarrow \infty} \|\bar{u}_{k,x}\|_{L^\infty(U)} \leq 1.$$

Consequently, one reaches the conclusion of Theorem 1.2 by summarizing the above discussion.

**Acknowledgment:** This project is partially supported by US Air Force Office of Scientific Research (AFOSR FA9550-10-1-0487), Natural Science Foundation of Jiangsu Province (BK 20130598), National Natural Science Foundation of China (NSFC 71273048, 71473036, 11471072), the Scientific Research Foundation for the Returned Overseas Chinese Scholars, Fundamental Research Funds for the Central Universities on the Field Research of Commercialization of Marriage between China and Vietnam (No. 2014B15214). This work is also supported by Open Research Fund Program of Jiangsu Key Laboratory of Engineering Mechanics, Southeast University (LEM16B06). In particular, the authors also express their deep gratitude to the referees for their careful reading and useful remarks.

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